On Fourier Series and Functions of Bounded Variation.

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§ 1. In a previous communication to the Society I have pointed out that the succession of constants obtained by multiplying together two successions of Fourier constants in the manner which naturally suggests itself is a succession of Fourier constants, and I have discussed the summability of the function with which the new constants are associated. We may express the matter in another way by saying that I have shown that the use of the Fourier constants of an even function $g(x)$ as convergence factors in the Fourier series of a function $f(x)$ changes the latter series into a series which is still a Fourier series, while the summability of the function which is associated with the new series is increased. The use of the Fourier constants of an odd function as convergence factors, on the other hand, has the effect of changing the allied series of the Fourier series of $f(x)$ into a Fourier series, even when the allied series is not itself a Fourier series.

It at once suggests itself that the former of the two statements in this form of the result is not the most that can be said. Indeed, the series, whose general term is $\cos nx$, and whose coefficients are accordingly unity, may clearly take the place of the Fourier series of $g(x)$, although it is not a Fourier series. On the other hand, it is the derived series of the Fourier series of a function of bounded variation, which is, moreover, odd. We are thus led to ask ourselves whether this is not the trivial case of a general theorem. In the present communication I propose to show, among other things, that the answer to this question is in the affirmative. The following theorems are, in fact, true:—

If the coefficients of the derived series of a Fourier series of an odd function of bounded variation be used as convergence factors, the Fourier series of a general summable function will remain the Fourier series of a summable function, while the degree of summability of the function will in general be unaltered. On the other hand, if the function of bounded variation used be even, the corresponding convergence factors obtained by deriving its Fourier series will have the effect of transforming the allied series of a Fourier series into the Fourier series of a function having the same degree of summability as the function corresponding to the original Fourier series.

These results appear to me to possess of themselves sufficient interest to justify my communicating them to the Society. Apart, however, from their intrinsic interest, the method by which I have been led to obtain them is, I think, at once suggestive and instructive. In germ this method has already been employed by Stieltjes. Properly handled and suitably developed, it constitutes a very effective tool in dealing with some of the finer distinctions in the Theory of Functions of a Real Variable. For this purpose, however, we require to introduce the notion of the summability of a function $f(x)$ with respect to a function of bounded variation $g(x)$, and not merely, as Stieltjes does, to employ an analytical expression, which may be interpreted as the Riemann integral of a continuous function taken with respect to a monotone function. Moreover, we require to have a theory of integration with respect to a function of bounded variation which shall correspond to that of integration with respect to the independent variable, such as I have exposed in a recent communication to the Society.* When this has been done we may integrate sequences and successions with respect to any assigned function of bounded variation, whether continuous or not, under conditions which are analogous to those in the known theory.

In particular, we have a theory of the integration of Fourier series term by term with respect to a function of bounded variation of the most general type. We are thus led to a variety of results of interest, and, among others, to those I have stated above. I have not attempted to give a complete account of the theory, but I have thought it well to take the opportunity to enunciate and give indications of proof of some of the connected results so obtained. Among these may be mentioned the one which states that if a trigonometrical series has the property that its integrated series converges everywhere boundedly, then term-by-term integration of the trigonometrical series is allowable after it has been multiplied by any function of bounded variation whatever, the sum of the latter integrated series being expressible in terms of the integral of the function of bounded variation with respect to the function which is the sum of the integrated trigonometrical series itself. Again, it appears that the coefficients of the derived series of the Fourier series of respectively an odd and an even function of bounded variation, when employed as convergence factors, change the Fourier series of a continuous function and its allied series into the Fourier series of continuous functions.

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* *Added June*, 1913.—In the order of ideas there indicated various slightly different but equivalent modes of treatment are possible. If we start with simple l and u functions and define, as we evidently may content ourselves with doing, their integrals with respect to a monotone increasing function $g(x)$, special attention must be paid to the discontinuities of $g(x)$, unless we hypothecate that the simple functions are so chosen that none of their discontinuities coincide with those of $g(x)$. We may also, if we please, start with the integral of a continuous function, using for this purpose the formula of Stieltjes; this corresponds to the treatment suggested in my paper, "On a New Method in the Theory of Integration," cited below. I am, by request, writing out a systematic account of these matters, and propose to present the paper containing it to the London Mathematical Society. All turns on the use of monotone sequences. Another, but far less intuitive form of treatment, has been briefly indicated by Lebesgue. He employs, but evidently with great reluctance, the process of change of the independent variable. See H. Lebesgue, 'Comptes Rendus,' 1909.

The theory of the summability of a function with respect to a function of bounded variation is, it will be noted, only part of a still larger theory. Even Stieltjes was led to consider integration with respect to a continuous function, owing to the necessity of formulating a theorem of integration by parts. In general, however, the generalisation in this direction involves difficulties of a nature analogous to those which arise when we are dealing with non-absolutely convergent integration with respect to the independent variable. I do not propose, therefore, in the present communication, to enter on these matters.

§ 2. It will not be necessary on this occasion to enter into details with regard to the theory of integration with respect to a function, and in particular with respect to a function of bounded variation. It will be evident that a function which is summable may, or may not, be summable with respect to an assigned function of bounded variation, and a sequence or succession which is integrable term by term may, or may not, remain integrable term by term when the integration is with respect to a function of bounded variation. The rules, however, which enable us to recognise such possibilities in the general case are usually completely analogous to the known ones in the special case. Thus we have merely to remark that the repeated integral with respect to two monotone increasing functions of a positive function is necessarily independent of the order in which the integration is performed, to see that, if $g(t)$ is any function of bounded variation whatever, and $f(t)$ is any summable function, the integral of $f(x+t)$ with respect to $g(t)$ certainly exists except for a set of values of x of content zero. Using Stieltjes' notation, and interpreting it in the extended sense above explained, we may accordingly write

$$\int_a^x dx \int f(x+t)\,dg(t) = \int \{F(x+t)-F(x-t)\}\,dg(t), \tag{1}$$

where $F(x)$ is an indefinite integral of $f(x)$.

We thus see that the right-hand side of (1) is an integral with respect to x, and that the inside integral on the left certainly exists, except at a set of content zero, while it constitutes a summable function of x.

§ 3. Again we see immediately that sequences that converge uniformly can certainly be integrated with respect to any function of bounded variation.

In fact let $f_n(x)$ be the general term of a sequence which approaches $f(x)$ uniformly. Then it is plain that $\int [f_n(t)-f(t)]\,dg(t)$ has, as $n\to\infty$, the unique limit zero. For we can find an n so that $|f_n(t)-f(t)|$ is less than e, for all points t in the interval of integration, and we may suppose $g(t)$ expressed as the difference of two positive monotone functions. Each of the corresponding integrals will then be less than a certain finite multiple of e, and is therefore as small as we please. Hence the limit of our integral, as $n\to\infty$, is zero.

But, more generally, it is sufficient that the sequence should converge boundedly for the same to be true. To see this we have, in fact, merely to retrace the steps of the reasoning by which, in a recent communication to the Society, I showed that the process is allowable when integration is taken in the new generalised sense there explained. *Mutatis mutandis* the whole argument applies.

§ 4. Let $f(x) \sim \Sigma(a_n \cos nx + b_n \sin nx)$, and consider the series

$$\Sigma\alpha_n(a_n \cos nx + b_n \sin nx), \tag{2}$$

where

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} n \sin nt\, g(t)\,dt, \tag{3}$$

$g(t)$ being an odd function of bounded variation, so that α_n is the typical coefficient of the trigonometrical series got by differentiating term by term the Fourier series of $g(t)$.

We are about to prove that the series (2) is the Fourier series of the following function—

$$\phi(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,dg(t). \tag{4}$$

We have by (4)

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\phi(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\cos nx\,dx\right)dg(t). \tag{5}$$

In fact, change of order of integration with respect to $g(t)$ and x is permitted when the factor cos nx is omitted, and is therefore, by a theorem, analogous to a known theorem in ordinary repeated integration, allowable when cos nx is present. Now the inside integral on the right-hand side of (5) is evidently equal to

$$a_n \cos nt + b_n \sin nt.$$

We can, in the further evaluation of the right-hand side of (5), almost without leaving the realm of Stieltjes' ideas, employ an obviously allowable generalisation of the ordinary theory of integration by parts. The term $b_n \sin nt$ evidently contributes nothing to the result, since $g(t)$ is odd ; and since $g(t)$ vanishes at the limits of integration* with respect to t, of the two parts involving a_n, one vanishes identically. Hence (5) becomes

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\phi(x)\cos nx\,dx = \frac{1}{\pi}a_n\int_{-\pi}^{\pi}g(t)\,n\sin nt\,dt = a_n\alpha_n. \tag{6}$$

Similarly
$$\frac{1}{\pi}\int_{-\pi}^{\pi}\phi(x)\sin nx = b_n\alpha_n. \tag{7}$$

The equations (6) and (7) show that the series (2) has the Fourier form, and that the corresponding function is $\phi(x)$, as was stated.

§ 5. Next consider the series

$$\Sigma\beta_n(-b_n\cos nx + a_n\sin nx), \tag{8}$$

where
$$\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi}n\cos nt\,h(t)\,dt, \tag{9}$$

$h(t)$ being an even function of bounded variation, so that $-\beta_n$ is the typical coefficient of the trigonometrical series got by differentiating term by term the Fourier series of $h(t)$.

We are about to prove that the series (8) is the Fourier series of the following function—

$$\psi(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)\,dh(t). \tag{10}$$

We have, as before,

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}\psi(x)\cos nx\,dx &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)\cos nx\,dx\right)dh(t)\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}(a_n\cos nt + b_n\sin nt)\,dh(t) = \frac{1}{\pi}\int_{-\pi}^{\pi}b_n\sin nt\,dh(t)\\ &= -b_n\beta_n,\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}\psi(x)\sin nx\,dx &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)\sin nx\,dx\right)dh(t)\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}(b_n\cos nt - a_n\sin nt)\,dh(t) = -\frac{1}{\pi}\int_{-\pi}^{\pi}a_n\sin nt\,dh(t)\\ &= a_n\beta_n.\end{aligned} \tag{12}$$

The equations (11) and (12) show that the series (8) has the Fourier form, and that the corresponding function is $\psi(x)$.

§ 6. We may next discuss the summability of the functions $\phi(x)$ and $\psi(x)$, and the mode in which this depends on the summability of $f(t)$. There is no difficulty in seeing that both these functions belong in general to the same class of functions as $f(x)$. If we consider in particular the case in which the $(1+p)$th power of $f(x)$ is summable, we have only to use the inequality

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* In fact, as we suppose $g(x)$ periodic and odd, its values when $x = -\pi$ and $x = +\pi$ must be both zero.

$$(\int u\,dv)^{1+p} \leq \int u^{1+p}dv\,.\,(\int dv)^p,$$

where u and v are any positive functions of the independent variable x.

More generally we may use the inequality

$$Q\left(\frac{\int u\,dv}{\int dv}\right) \leq \frac{\int Q(u)\,dv}{\int dv},$$

where $Q(t)$ is the indefinite integral of any positive function whose differential coefficient is positive (>0).* This inequality may evidently be proved in a manner analogous to that used in proving the corresponding inequality† when v is itself the independent variable, with whose limits of integration we are concerned.

Putting then $u = f(x+t)$ and v equal to the total variation of $g(t)$, or $h(t)$ as the case may be, and taking t to be the independent variable, the required results immediately follow.

§ 7. It may also be remarked that the corresponding reasoning shows that the effect of integrating the Fourier series of a function of bounded variation, and its allied series, with respect to an odd function of bounded variation and an even function of bounded variation respectively is, in the former case, to preserve the character of being the Fourier series of such a function; and, in the latter case, to transform the allied series into such a Fourier series. Or, as we may otherwise express it, the convergence factors α_n and β_n have these effects respectively.

§ 8. If, on the other hand, we perform the process of integration with respect to a function of bounded variation on the Fourier series of a continuous function, we easily prove that the following result holds good:—

If α_n and β_n are the typical coefficients of the derived series of the Fourier series of respectively an odd and an even function of bounded variation, then the convergence factor α_n transforms the Fourier series of a continuous function into the Fourier series of a continuous function, while the convergence factor β_n changes the allied series of the Fourier series of such a function into the Fourier series of a continuous function.

In fact, the argument is precisely the same as that employed in §§ 4 and 5, and shows that the functions with which these series, as Fourier series, are associated are respectively

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* "On Classes of Summable Functions and their Fourier Series," 'Roy. Soc. Proc.,' 1912, A, vol. 87, p. 227.

† "On the New Theory of Integration," 'Roy. Soc. Proc.,' 1912, A, vol. 88; see also "On a New Method in the Theory of Integration," 'Lond. Math. Soc. Proc.,' 1910, Ser. 2, vol. 9, pp. 15–50.

$$\phi(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, dg(t) \quad \text{and} \quad \psi(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, dh(t),$$

where g and h are respectively the odd and even functions of bounded variation. The theorem stated follows immediately from the known property of a continuous function of being uniformly continuous. We may indeed find a quantity η, independent of t (and of x), such that for this and all smaller values,

$$| f(x+t+\eta) - f(x+t) |$$

is less than a fixed quantity e, as small as we please.

Hence, for $0 \leq \delta \leq \eta$,

$$| \phi(x+\delta) - \phi(x) | \leq \frac{e}{\pi}\int_{-\pi}^{\pi} | dg(t) | ,$$

which, as the total variation of $g(t)$ is finite, proves the continuity of $\phi(x)$. Similarly, $\psi(x)$ is seen to be continuous.

Or, more simply, we may remark that the sequence $f(x+t+h)$ converges uniformly to $f(x+t)$ as $h \rightarrow 0$, whence the required result follows by § 3.

§ 9. Finally we shall prove the following theorem, which, it will be seen, is of a slightly different type.

If a_n and b_n are the typical Fourier constants of a function $f(x)$ of bounded variation, and A_n and B_n the typical coefficients of a trigonometrical series whose integrated series converges boundedly to $G(x)$, then the series whose general term is

$$(a_n A_n + b_n B_n) \cos nx - (a_n B_n - b_n A_n) \sin nx$$

converges boundedly and has for sum

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, dG(t).$$

To prove this we have only to employ the theorem to which allusion has been made in the last few lines of § 3.

In fact, since the series whose general term is

$$\frac{1}{n}(A_n \sin nx - B_n \cos nx)$$

converges boundedly to $G(x)$, it is the Fourier series of $G(x)$, and to prove the result we have merely to integrate term by term the Fourier series of $G(t-x)$ with respect to $f(t)$. We shall then get the right-hand side of the equation to be proved with its sign changed on the right of our equation, while on the left we shall have the expression $\frac{1}{\pi}\int_{-\pi}^{\pi} G(t-x)\,df(t)$. Carrying out the integration in this last expression by parts, and remembering that G and f are both periodic, and in particular are supposed at have at the one limit of integration the same values as at the other limit, the required result follows.

§ 10. The result just obtained appears to be not without importance, more especially in view of the circumstances that, though it is not known when we may safely assert that the allied series of a Fourier series is itself a Fourier series, we do know that the integrated series of the allied series of a Fourier series converges boundedly, and even uniformly, in a large variety of cases.

It will be seen that all has turned in the present theory on the fact that a function of bounded variation is not necessarily an integral. In the cases which most naturally present themselves the function is only not an integral because it has one or more finite discontinuities. In the best known example of the Fourier series of a function of bounded variation, viz. the series whose general term is $\sin nx/n$, this is precisely what occurs. We may verify our results in this trivial case by showing that the integral of a function $f(t)$ which has only discontinuities of the first kind with respect to the function $g(t)$ of bounded variation which is the sum of the series $\Sigma \sin nx/n$, is the original function, or, more precisely,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,dg(t) = \tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

The difference in the behaviour of a Fourier series and its allied series corresponds to the fact that the series whose general term is $\cos nx/n$ is not the Fourier series of a function of bounded variation. It would be of interest to obtain the simplest cosine series having the property of being the Fourier series of a discontinuous function of bounded variation. A knowledge of such a series would give us a number of interesting results, which would follow as immediate consequences of the reasoning we have here employed.